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ON THE QUADRATIC NORMALITY AND THE TRIPLE CURVE OF THREE DIMENSIONAL SUBVARIETIES OF $\mathbb{P}^5$

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Abstract. A well-known conjecture asserts that smooth threefolds $X \subset \mathbb{P}^5$ are quadratically normal with the only exception of the Palatini scroll. As a corollary of a more general statement we obtain the following result, which is related to the previous conjecture: If $X \subset \mathbb{P}^5$ is not quadratically normal, then its triple curve is reducible. Similar results are also given for higher dimensional varieties.

1. Introduction

Let $X \subset \mathbb{P}^r$ be a reduced irreducible complex projective subvariety. X is said to be k-normal if the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(X, \mathcal{O}_X(k))$ is surjective or, equivalently, if $H^1(\mathbb{P}^r, \mathcal{I}_X(k)) = 0$.

A conjecture of Peskine and Van de Ven states that a smooth threefold $X \subset \mathbb{P}^5$ is 2-normal unless it is the Palatini scroll (see, for instance, [10, Problem 5]). We remark that smoothness cannot be dropped according to [1]. On the other hand, the Palatini scroll is also the only known example of smooth threefold X in $\mathbb{P}^5$ with reducible triple curve (cf. [6]).

Let us define the triple locus. A line $L \subset \mathbb{P}^5$ is said to be *k-secant* to $X \subset \mathbb{P}^5$ (respectively, *strict* k-secant) if the scheme-theoretic intersection $X \cap L$ has length at least k (respectively, equal to k). We define the *triple locus* $\Gamma_P \subset X$ as the subvariety of points contained in a 3-secant line to X passing through a point $P \in \mathbb{P}^5$. We say that the triple locus of X is irreducible if Γ_P is irreducible for a general $P \in \mathbb{P}^5$, and we say that an irreducible triple locus is not quadruple if the general 3-secant line to X passing through P is a strict 3-secant line.

The main aim of this note is to prove the following result:

Theorem 1. *Let $X \subset \mathbb{P}^5$ be an integral subvariety of dimension three. If the triple locus of X is non-empty, irreducible and not quadruple, then X is 2-normal.*

Moreover, we say that L is a *true* k-secant line if $X_{\mathrm{sm}} \cap L$ has length at least k, where $X_{\mathrm{sm}} \subset X$ is the open subset of the smooth points. We also define the *true triple locus* $\gamma_P \subset X$ as the subvariety obtained by taking the closure of the set of points contained in a true 3-secant line to X passing through P, and we say, as before, that the true triple locus of X is irreducible if γ_P is irreducible for a general $P \in \mathbb{P}^5$.

We prove in Lemma 4 that the true triple locus $\gamma_P \subset X$ is at most 1-dimensional for a general $P \in \mathbb{P}^5$. Furthermore, if $\gamma_P \subset X$ is actually 1-dimensional we show in Lemma 5 that all but a finite number of true 3-secant lines passing through P are strict. Both lemmas easily follow from the generalization of the classical Trisecant

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Lemma given in [9, Theorem 1]. Therefore, we finally define the *true triple curve of* X as the 1-dimensional component of γ_P. Then we can slightly modify the proof of Theorem 1 to obtain the following:

Corollary 2. *Let $X \subset \mathbb{P}^5$ be an integral subvariety of dimension three. If the true triple curve of X is non-empty and irreducible, then X is 2-normal.*

If moreover X is smooth then its (true) triple curve is actually non-empty unless $X \subset \mathbb{P}^5$ is contained in a quadric hypersurface (see [5]), so Corollary 2 yields:

Corollary 3. *Let $X \subset \mathbb{P}^5$ be a smooth threefold. If X is not 2-normal, then the triple curve of X is reducible.*

Hence in order to prove the conjecture of Peskine and Van de Ven, it is enough to prove the following:

Conjecture 1. *The only smooth threefold $X \subset \mathbb{P}^5$ with reducible triple curve is the Palatini scroll.*

Finally, in Section 4, Theorem 1 and its corollaries are generalized to varieties of higher dimension.

2. A geometric point of view

To begin with, we present a geometric proof of the fact that if X in $\mathbb{P}^4$ is an integral surface with irreducible double curve, then X is linearly normal. This follows as a consequence of the classical characterizations of the Veronese surface given by Severi [11] and Franchetta [3], where, in the latter, X is supposed to have as singularities at most a finite number of *improper double points* (*i.e.* the tangent cone is the union of two planes in general position). This fact was proved in [7] under Franchetta's assumption by using the monoidal construction.

The double locus $\Delta_Q \subset X$ is defined as the set of points contained in a 2-secant line to X passing through a general point $Q \in \mathbb{P}^4$. This locus is 1-dimensional whenever $X \subset \mathbb{P}^4$ is non-degenerate, but it is not necessarily equidimensional since smoothness of X is not required. Furthermore, if Δ_Q is irreducible the general 2-secant line to X passing through Q is a strict 2-secant line by the well-known Trisecant Lemma.

Fact 1. *Let $X \subset \mathbb{P}^4$ be an integral surface. If the double locus of X is irreducible, then X is linearly normal.*

Proof. We can assume $X \subset \mathbb{P}^4$ non-degenerate. To get a contradiction, let us suppose that $X \subset \mathbb{P}^4$ is not linearly normal. Then there exists a linear projection

$$\pi_P : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$$

from a point $P \in \mathbb{P}^5$ and a non-degenerate surface $X' \subset \mathbb{P}^5$ such that the restriction map

$$\pi_P\,|_{X'} : X' \to X$$

is an isomorphism. In particular, P does not belong to any 2-secant line to X'. Let $SX' \subset \mathbb{P}^5$ denote the 2-secant variety defined as the locus of points in $\mathbb{P}^5$ contained in a 2-secant line to X'. We have $\dim(SX') \leq 4$ since $\pi_P\,|_{X'} : X' \to X$ is an isomorphism. On the other hand $\dim(SX') \geq 4$, since a 3-dimensional variety containing a 4-dimensional family of lines is necessarily $\mathbb{P}^3$ and, in that case, $X' \subset \mathbb{P}^5$ would be degenerate. Hence $\dim(SX') = 4$. Let $W \subset SX'$ denote the irreducible 4-dimensional component corresponding to the closure of the union of lines joining two different points of X', and let $d > 1$ denote its degree in $\mathbb{P}^5$. Note that through any point of W there pass infinitely many 2-secant lines to X'.

Let $Q \in \mathbb{P}^4$ be a general point and let $\pi_P^{-1}(Q) \cap W = \{Q_1, \dots, Q_d\} \subset \mathbb{P}^5$. We have $Q_i \neq Q_j$ for $i \neq j$ and $Q_i \notin X'$, since Q is general. For $i \in \{1, \dots, d\}$ we denote by $\Delta'_{Q_i} \subset X'$ the double locus corresponding to the points contained in a 2-secant line to X' passing through Q_i. Since $Q_i \notin X'$ the family of 2-secant lines to X' passing though Q_i cannot be 2-dimensional, so Δ'_{Q_i} is actually 1-dimensional.

We claim that $\Delta'_{Q_{i_0}} \neq \Delta'_{Q_j}$ for some $i_0 \in \{1, \dots, d\}$ and every $j \neq i_0$. First, we remark that if $L \subset \mathbb{P}^4$ is a strict 2-secant line to X, then there exists a unique 2-secant line $L' \subset \mathbb{P}^5$ to X' such that $\pi_P(L') = L$. This is obvious if L intersects X in two different points. If $L \cap X$ is a double point then $\pi_P\mid_{X'}^{-1}(L \cap X)$ is a double point, whence there exists a unique line $L' \subset \mathbb{P}^5$ such that $L' \cap X' = \pi_P\mid_{X'}^{-1}(L \cap X)$. Let us prove the claim. Since $Q \in \mathbb{P}^4$ is a general point, there exists a general point $Q_{i_0} \in W$ such that $\pi_P(Q_{i_0}) = Q$. Assume by contradiction that $\Delta'_{Q_{i_0}} = \Delta'_{Q_j}$, and let $x' \in \Delta'_{Q_{i_0}}$ be a general point (of course $\Delta'_{Q_{i_0}}$ may be reducible, but in this case so is Δ_Q). Then the lines $L'_{i_0} := \langle x', Q_{i_0} \rangle$ and $L'_j := \langle x', Q_j \rangle$ are 2-secant lines to $X' \subset \mathbb{P}^5$ such that $\pi_P(L'_{i_0}) = \pi_P(L'_j) = \langle \pi_P(x'), Q \rangle =: L$ is a 2-secant line to $X \subset \mathbb{P}^4$. Moreover $x' \in X'$ is a general point since $Q_{i_0} \in W$ is a general point. Then $\pi_P(x') \in X$ is also a general point, and hence $L \subset \mathbb{P}^4$ is a strict 2-secant line to X. This contradicts the previous remark, since $\pi_P(L'_{i_0}) = \pi_P(L'_j) = L$.

Since $\pi_P\mid_{X'} \colon X' \to X$ is an isomorphism we have $\pi_P(\Delta'_{Q_{i_0}}) \neq \pi_P(\Delta'_{Q_j})$ for every $j \neq i_0$. Therefore $\Delta_Q \subset X$ has at least two irreducible components, contradicting the hypothesis. □

3. Proof of Theorem 1

We obtain our main result inspired by the geometric proof of Fact 1.

Proof of Theorem 1. We divide the proof in three parts:

Step 1: *Set up.* Let us consider the Veronese embedding

$$v_2 : \mathbb{P}^5 \to v_2(\mathbb{P}^5) \subset \mathbb{P}^{20}$$

given by the complete linear system of quadrics in $\mathbb{P}^5$ and let

$$v_2|_X : X \to v_2(X) \subset \mathbb{P}^{20}$$

denote its restriction to X. Since the triple locus of $X \subset \mathbb{P}^5$ is non-empty we get $H^0(\mathbb{P}^5, \mathcal{I}_X(2)) = 0$, whence $v_2(X) \subset \mathbb{P}^{20}$ is non-degenerate.

To get a contradiction we assume that $X \subset \mathbb{P}^5$ is not 2-normal. This is equivalent to suppose that $v_2(X) \subset \mathbb{P}^{20}$ is not linearly normal. Then there exists a point $P \in \mathbb{P}^{21}$ and a non-degenerate threefold $X' \subset \mathbb{P}^{21}$ such that the linear projection

$$\pi_P : \mathbb{P}^{21} \dashrightarrow \mathbb{P}^{20}$$

from P induces by restriction an isomorphism

$$\pi_P|_{X'} : X' \to v_2(X).$$

Step 2: *The variety of* 3*-secant conics.* We denote by $C_P(v_2(\mathbb{P}^5)) \subset \mathbb{P}^{21}$ the 6-dimensional cone of vertex P over $v_2(\mathbb{P}^5) \subset \mathbb{P}^{20}$. Let $\mathcal{H}$ be the Hilbert scheme of conics in $\mathbb{P}^{21}$. We say that a conic $C \in \mathcal{H}$ is 3-secant to $X' \subset \mathbb{P}^{21}$ if the scheme-theoretic intersection $C \cap X'$ has length at least three. Furthermore, C is said to be a strict 3-secant conic if $\mathrm{length}(C \cap X') = 3$. Let $V \subset C_P(v_2(\mathbb{P}^5))$ denote the subvariety of points lying on a 3-secant conic to X' which is contained in $C_P(v_2(\mathbb{P}^5))$. We remark that V plays the role of the 2-secant variety $SX' \subset \mathbb{P}^5$ in the proof of Fact 1.

We claim that $P \notin V$. Otherwise, there exists a (maybe reducible or non-reduced) 3-secant conic C to X' passing through P such that $C \subset C_P(v_2(\mathbb{P}^5))$. Let $\mathbb{P}^2_C \subset \mathbb{P}^{21}$ denote the linear span of C. Then, $\pi_P|_{X'} : X' \to v_2(X)$ being

an isomorphism, we get that $\pi_P(\mathbb{P}^2_C) \subset \mathbb{P}^{20}$ is a 3-secant line to $v_2(X)$. This is impossible since $v_2(\mathbb{P}^5)$, and hence $v_2(X)$, has no 3-secant lines.

It follows that $\dim V < 6$. Let us see that $\dim V = 5$ proving that $\pi_P(V)$ contains $v_2(\mathbb{P}^5)$. Since the triple locus of $X \subset \mathbb{P}^5$ is non-empty, there exists an irreducible family $\{L_z\}_{z\in Z}$ of 3-secant lines to X that fill up $\mathbb{P}^5$. We choose Z of maximal dimension satisfying this property, so a general L_z is a strict 3-secant line to X by hypothesis. It is enough to prove that a strict 3-secant conic $v_2(L_z)$ to $v_2(X)$ can be uniquely lifted to a 3-secant conic C_z to X' such that $\pi_P(C_z) = v_2(L_z)$. Let ξ be the scheme-theoretic intersection $L_z \cap X$ of length three. Then $v_2(L_z)$ is a strict 3-secant conic to $v_2(X)$. We now consider the subscheme $\xi' := \pi_P\mid_{X'}^{-1}(v_2(\xi))$ of X' of length three. We remark that there exists a unique plane $\mathbb{P}^2_{\xi'}$ such that $\mathbb{P}^2_{\xi'} \cap X' = \xi'$. Then we define the conic $C_z \subset \mathbb{P}^2_{\xi'}$ as the intersection of the quadric cone $C_P(v_2(L_z))$ and $\mathbb{P}^2_{\xi'}$. Moreover, $\pi_P(C_z) = v_2(L_z)$ since $P \notin \mathbb{P}^2_{\xi'}$ by the same reason of the above claim. This proves $v_2(\mathbb{P}^5) \subset \pi_P(V)$, whence $\pi_P(V) = v_2(\mathbb{P}^5)$.

Step 3: *End of the proof.* Let $V' \subset V$ denote the irreducible 5-dimensional component arising from the union of the conics corresponding to the family Z. The general point $x \in X$ is contained in a 3-secant line to X since the family $\{L_z\}$ fills up $\mathbb{P}^5$. Hence $v_2(x) \in v_2(X)$ is contained in a 3-secant conic to $v_2(X)$, so $x' := \pi_P^{-1}|_{X'}(v_2(x)) \in X'$ is a general point contained in a 3-secant conic to X'. Therefore $X' \subset V'$. Note that $V' \subset \mathbb{P}^{21}$ is non-degenerate since $X' \subset \mathbb{P}^{21}$ is non-degenerate and $X' \subset V'$. Hence the induced linear projection

$$\pi_P\mid_{V'} \colon V' \to v_2(\mathbb{P}^5)$$

cannot be an isomorphism since $v_2(\mathbb{P}^5) \subset \mathbb{P}^{20}$ is linearly normal. We claim that moreover $\pi_P\mid_{V'} \colon V' \to v_2(\mathbb{P}^5)$ cannot be a birational morphism. Otherwise, by Zariski's Main Theorem, all its fibres should be connected. In particular, if we consider two points with the same image, the corresponding fibre should be the whole line through P, which is impossible since $P \notin V'$.

We deduce that $\pi_P\mid_{V'} \colon V' \to v_2(\mathbb{P}^5)$ is a morphism of degree $d > 1$ by the previous claim.

Let $Q \in \mathbb{P}^5$ be a general point and let $\Gamma_Q \subset X$ denote the corresponding triple locus. Consider $v_2(Q) \in v_2(\mathbb{P}^5)$ and let

$$\pi_P^{-1}(v_2(Q)) \cap V' = \{Q_1, \ldots, Q_d\} \subset \mathbb{P}^{21}.$$

We remark that $Q_i \neq Q_j$ for $i \neq j$ and $Q_i \notin X'$, since $Q \in \mathbb{P}^5$ is a general point. For $i \in \{1, \ldots, d\}$ we denote by $\Gamma'_{Q_i} \subset X'$ the locus of points lying on a 3-secant conic to X' passing through Q_i and contained in V.

We now check that $\Gamma'_{Q_{i_0}} \neq \Gamma'_{Q_j}$ for some $i_0 \in \{1, \ldots, d\}$ and every $j \neq i_0$. Since $Q \in \mathbb{P}^5$ is a general point, there exists a general point $Q_{i_0} \in V'$ such that $\pi_P(Q_{i_0}) = v_2(Q)$. Assume to the contrary that $\Gamma'_{Q_{i_0}} = \Gamma'_{Q_j}$ and let $x' \in \Gamma'_{Q_{i_0}}$ be a general point (as in Fact 1, $\Gamma'_{Q_{i_0}}$ may be reducible but in this case so is Γ_Q). Then there exist two 3-secant conics C_{i_0} and C_j, passing through x' and contained in V', such that $Q_{i_0} \in C_{i_0}$ and $Q_j \in C_j$. Let us see that $\pi_P(C_{i_0}) = \pi_P(C_j)$. Consider the line $L := \langle x, Q\rangle \subset \mathbb{P}^5$, where $x = v_2^{-1}(\pi_P(x')) \in X$. Then $v_2(L)$ is the only 3-secant conic to $v_2(X)$ joining $v_2(Q)$ and $\pi_P(x')$ and contained in $v_2(\mathbb{P}^5)$, so necessarily $\pi_P(C_i) = \pi_P(C_j) = v_2(L)$. Moreover L is a strict 3-secant line to $X \subset \mathbb{P}^5$ since $x' \in X'$, and hence $\pi_P(x') \in v_2(X)$, is a general point. Therefore $v_2(L)$ is a strict 3-secant conic to $v_2(X)$, so it can be uniquely lifted to a 3-secant conic to X' contained in V, as we showed in Step 2. This yields a contradiction since $\pi_P(C_i) = \pi_P(C_j) = v_2(L)$.

It follows from $\Gamma'_{Q_{i_0}} \neq \Gamma'_{Q_j}$ that also $\pi_P(\Gamma'_{Q_{i_0}}) \neq \pi_P(\Gamma'_{Q_j})$, since $\pi_P|_{X'} : X' \to v_2(X)$ is an isomorphism. So we finally deduce that $\Gamma_Q \subset X$ has at least two irreducible components, contradicting the hypothesis. □

Remark 1. The converse of Theorem 1 does not hold since there exist 2-normal cones $X \subset \mathbb{P}^5$ with vertex a point over a surface whose triple locus consists of a union of rulings of X.

We pass to prove Corollary 2. We first show that the true triple locus of X is at most 1-dimensional:

Lemma 4. *Let $X \subset \mathbb{P}^{n+2}$ be a subvariety of dimension $n \geq 1$. Then the family of true n-secant lines to X passing through a general point $P \in \mathbb{P}^{n+2}$ is at most 1-dimensional.*

Proof. Assume that the family of true n-secant lines to X passing through a general $P \in \mathbb{P}^{n+2}$ is at least 2-dimensional. Let $Y = X \cap H_1 \cap H_2$ be the intersection of X with two general hyperplanes of $\mathbb{P}^{n+2}$. Then the union of the true n-secant lines to Y fill up $\mathbb{P}^n$, contradicting [9, Theorem 1]. □

Moreover, if the true triple locus is 1-dimensional then there exists a 5-dimensional family Z of true 3-secant lines to X filling up $\mathbb{P}^5$. In this case the general element of Z is a strict 3-secant line, as the following lemma shows:

Lemma 5. *Let $X \subset \mathbb{P}^{n+2}$ be a subvariety of dimension $n \geq 1$. Let Z be an irreducible $(n+2)$-dimensional variety parametrising a family $\{L_z\}$ of true n-secant lines to $X \subset \mathbb{P}^{n+2}$ whose points fill up $\mathbb{P}^{n+2}$. Then a general L_z is a strict n-secant line to X.*

Proof. Assume to the contrary that a general L_z is an $(n+1)$-secant line to X. Let $Y = X \cap H \subset \mathbb{P}^{n+1}$ be a hyperplane section of $X \subset \mathbb{P}^{n+2}$. Then a general $P \in \mathbb{P}^{n+1}$ is contained in a true $(n+1)$-secant line to Y since P is contained in infinitely many $(n+1)$-secant lines to X, contradicting again [9, Theorem 1]. □

Proof of Corollary 2. If the true triple curve of X is non-empty and irreducible, we can repeat the proof of Theorem 1 considering in Step 2 a 5-dimensional family $\{L_z\}$ of true 3-secant lines filling up $\mathbb{P}^5$, whose general element is a strict 3-secant line by Lemma 5. □

We now show that Corollary 3 easily follows from Corollary 2:

Proof of Corollary 3. If the triple curve of X is non-empty we apply Corollary 2. On the other side, if the triple curve of X is empty then $H^0(\mathbb{P}^5, \mathcal{I}_X(2)) \neq 0$ by [5, Lemma 3.3 and Theorem 3.4 (a)] since X is smooth. Therefore $X \subset \mathbb{P}^5$ is projectively normal as a consequence of [4, Theorem 1]. □

4. Final remarks

We would like to stress that the proof of Theorem 1 can be extended to obtain a similar result for codimension two subvarieties $X \subset \mathbb{P}^{n+2}$ of higher dimension:

Theorem 6. *Let $X \subset \mathbb{P}^{n+2}$ be an integral subvariety of dimension $n \geq 4$. If the n-tuple locus of X is non-empty, irreducible and not $(n+1)$-tuple, then X is $(n-1)$-normal.*

Proof. (Sketch) If $X \subset \mathbb{P}^{n+2}$ is not $(n-1)$-normal and the n-tuple locus of X is non-empty, then there exists a point P and a non-degenerate embedding $X' \subset \mathbb{P}^{\binom{2n+1}{n-1}}$ such that the linear projection from P to $v_{n-1}(X) \subset \mathbb{P}^{\binom{2n+1}{n-1}-1}$ is an isomorphism. Then we can repeat the proof of Theorem 1, having in mind that a strict n-secant

rational normal curve $v_{n-1}(L)$ of degree $n-1$ to $v_{n-1}(X)$, can be uniquely lifted to an n-secant rational normal curve of degree $n-1$ to X' contained in the cone $C_P(v_{n-1}(\mathbb{P}^{n+2}))$. □

In view of Lemma 4, we can also define the true n-tuple curve of $X \subset \mathbb{P}^{n+2}$ as the 1-dimensional component of the true n-tuple locus. Then the last assumption of Theorem 6 holds by Lemma 5, and we can also generalize Corollary 2 in the following way:

Corollary 7. *Let $X \subset \mathbb{P}^{n+2}$ be an integral subvariety of dimension $n \geq 4$. If the true n-tuple curve of X is non-empty and irreducible, then X is $(n-1)$-normal.*

According to Hartshorne's Conjecture, smooth subvarieties $X \subset \mathbb{P}^{n+2}$ of dimension $n \geq 4$ are expected to be complete intersections (and hence projectively normal). However, we remark that Corollary 3 can be also extended to higher dimensions thanks to [8]:

Corollary 8. *Let $X \subset \mathbb{P}^{n+2}$ be a smooth n-fold, $n \geq 4$. If X is not $(n-1)$-normal, then the n-tuple curve of X is reducible.*

Proof. If the n-tuple curve of X is non-empty the result follows from Corollary 7. On the other hand, if the n-tuple curve of X is empty then it follows from [8] that $X \subset \mathbb{P}^{n+2}$ is a complete intersection. For the reader's convenience, we explain this point in detail. Let $Y = X \cap H$ be a general hyperplane section. It follows from Barth and Larsen's theorems that $X \subset \mathbb{P}^{n+2}$ is subcanonical, whence $Y \subset \mathbb{P}^{n+1}$ is also subcanonical. Note that the n-tuple locus of Y is empty since the n-tuple locus of X is at most 0-dimensional. Then we apply [8, Proposition] to $Y \subset \mathbb{P}^{n+1}$. Since $\Sigma_n = \emptyset$, it follows that $e(k) := d - k\nu + k^2 = 0$ for some $k \in \{1, \dots, n-1\}$, where d is the degree of $Y \subset \mathbb{P}^{n+1}$ and $\bigwedge N_{Y/\mathbb{P}^{n+1}} \cong \mathcal{O}_Y(\nu)$. Then $\nu = \frac{d}{k} + k$. We remark that [8, Theorem] holds under the weaker hypothesis $\nu \geq \frac{d}{\alpha} + \alpha$ for some $\alpha \in (0, n-1]$, as the author pointed out in the beginning of the proof. Therefore $Y \subset \mathbb{P}^{n+1}$ (and hence $X \subset \mathbb{P}^{n+2}$) is a complete intersection. □

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